Minimal Model Program Learning Seminar.

Week 11:

- Restriction Theorem,
- Subadditivity Theorem.

(X, Δ) log pair, V linear system.

$$K_Y + \Gamma = f^*(K_X + \Delta) + E$$

$F = \mathrm{Fix}(f^* V)$, then we define the multiplier ideal

$$\mathcal{J}((X, \Delta); cV) = \mathcal{J}_{\Delta, c \cdot V} := f_* \mathcal{O}_Y(E - \lfloor cF \rfloor).$$

Lemma: The definition does not depend on the chosen log resolution.

Lemma: The more sing (X, Δ), D are and the larger c is, the deeper the ideal $\mathcal{J}_{\Delta, c \cdot D}$ is.

Theorem (Nadel vanishing): (X, Δ) quasi-projective log pair. N Cartier so that $N-D$ is ample for $D \geq 0$ $\mathbb{Q}$-Cartier. Then

$$H^i(X, \mathcal{J}_{\Delta,D}(K_X+\Delta+N)) = 0 \quad \text{for } i>0$$

In particular, if S is a component of Δ which appears with coefficient one, then

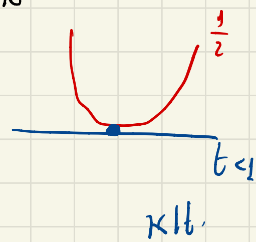
$$H^0(X, \mathcal{J}_{\Delta,D}(K_X+\Delta+N)) \quad \text{surjects onto}$$

$$H^0(S, \mathcal{J}_{(\Delta-S)|_S, D|_S}(K_X+\Delta+N)).$$

Theorem (Restriction theorem): X smooth variety, $D \geq 0$ $\mathbb{Q}$-divisor and $H \subseteq X$ smooth hypersurface not contained in the support of D. Then, there is an inclusion.

$$\mathcal{J}(H, D_H) \subseteq \mathcal{J}(X, D)_H := \mathcal{J}(X, D) \cdot \mathcal{O}_H.$$

Example: $X = \mathbb{C}^2_{s,t}$, H be the t-axis and $C = \{s - t^2 = 0\}$, $D = \frac{1}{2} C$.



$\mathcal{J}(X, D) = \mathcal{O}_X$ $\qquad$ $\mathcal{J}(X, D) \cdot \mathcal{O}_H = \mathcal{O}_H$

$(\mathbb{C}^2, \frac{1}{2} C)$ is log smooth with coeff $(\frac{1}{2} C) < 1$, which means is klt

$D_H = \operatorname{div}\langle t \rangle$, hence $\mathcal{J}(H, D_H) = \mathcal{J}(\mathbb{C}, (0)) = \langle t \rangle$.

$$\mathcal{J}(H, D_H) = \langle t \rangle \subseteq \mathcal{J}(X, D) \cdot \mathcal{O}_H = \mathbb{C}[t].$$

Remark: $\mathcal{J}(H, D_H) \subseteq \mathcal{J}(X, D + (1-t) H) \cdot \mathcal{O}_H$ for every $0 < t \leq 1$.

Proof: $\mu : X' \longrightarrow X$ log resolution of $(X, D+H)$.

Write $\quad \mu^* H = H' + \sum_i a_j E_j \quad (a_j \geq 0) \qquad (1)$

if H is general in a free linear system as in the example, all the a_j's are 0.

$$\begin{tikzcd} H' \arrow[r, hook] \arrow[d, "\mu_H"'] & X' \arrow[d, "\mu"] \\ H \arrow[r, hook] & X \end{tikzcd}$$

log resolution for (H, D_H)

$$\mathcal{J}(X, D) = \mu_* \mathcal{O}_{X'} (K_{X'/X} - [\mu^* D])$$

$$\mathcal{J}(H, D_H) = \mu_{H*} \mathcal{O}_{H'} (K_{H'/H} - [\mu_H^* D_H])$$

We have that $\quad [\mu_H^* D_H] = [\mu^* D]|_{H'}$.

By adjunction: $\quad K_{H'} \sim (K_{X'} + H')|_{H'} \qquad (2)$

$\qquad\qquad\qquad\quad K_H \sim (K_X + H)|_H \qquad (3)$

From (1), (2) and (3), we conclude that

$$K_{H'/H} = (K_{X'/X} - \textstyle\sum_i a_j E_j)|_{H'}. \qquad (4)$$

Define $\quad B := K_{X'/X} - [\mu^* D] - \sum_i a_j E_j.$

$$B|_{H'} = K_{H'/H} - [\mu_H^* D_H]$$

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$$B|_{H'} = K_{H'/H} - [\mu_H^* D_H]$$

Then, we have that $\mathcal{J}(H, D_H) = \mu_{H*} \mathcal{O}_{H'}(B)$.

On the other hand. diff is eff and μ-ex.

$$\mu_* \mathcal{O}_{X'}(B) \subseteq \mu_* \mathcal{O}_{X'}(K_{X'/X} - [\mu^* D]) = \mathcal{J}(X, D).$$
$$=$$

It suffices to prove

we need to prove this equality.

$$\mu_{H*} \mathcal{O}_{H'}(B) = \mu_* \mathcal{O}_{X'}(B) \cdot \mathcal{O}_H :=$$

$$\operatorname{Im}(\mu_* \mathcal{O}_{X'}(B) \hookrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_H).$$

Observe that $B - H' = K_{X'/X} - [\mu^*(D+H)]$.

By local vanishing, we obtain

$$R^1 \mu_* \mathcal{O}_{X'}(B - H') = 0.$$

Then, the proof follows by pushing forward wrt μ the seq:

$$0 \longrightarrow \mathcal{O}_{X'}(B - H') \xrightarrow{\cdot H'} \mathcal{O}_{X'}(B) \longrightarrow \mathcal{O}_{H'}(B) \longrightarrow 0$$

$\square$

Example: Let $|V|$ be a free linear system and $H \in |V|$ a general element. Then, we have that

$$\mathcal{J}(H, D_H) = \mathcal{J}(X, D)_H$$

Corollary (Inversion of Adjunction I):

In the setting of the restriction theorem

If we fix a point $x \in H$ and suppose that $\mathcal{J}(H, D_H)_x = \mathcal{O}_{H,x}$.

Then $\mathcal{J}(X, D + (1-t)H)_x = \mathcal{O}_{X,x}$.

For any rational number $0 < t \leq 1$

Equivalently, if (H, D_H) is klt near x, then $(X, D + (1-t)H)$ is klt near x. for $0 < t \leq 1$ (for $t = 0$ we have).

Remark: (H, D_H) is klt, then $(X, D+H)$ is plt

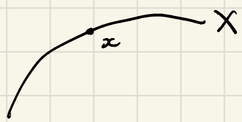
Proposition: $D \geq 0$ be a $\mathbb{Q}$-divisor on X smooth. $x \in X$ a point for which $\mathrm{mult}_x D < 1$. Then $\mathcal{J}(D)_x = \mathcal{O}_{X,x}$

Remark: $D = \sum_i a_i D_i$ with D_i Cartier $a_i \geq 0$

$$\mathrm{mult}_x D = \sum_{i=1}^{r} a_i \,\mathrm{mult}_x D_i.$$

Proof of prop: We proceed by induction on the dimension

$D = qx$



$$\mathcal{J}(X, qx)_x = \mathcal{O}_{X,x} \quad q < 1.$$

$$\mathcal{J}(X, x)_x = \mathfrak{m}_x.$$

$$\mathcal{J}(X, kx)_x = \mathfrak{m}_x^{\lfloor k \rfloor}$$

Hence, the statement is true in dimension one.

$H \subseteq X$ smooth hypersurface passing through x, we can take it with the following properties:

$\forall D_i$ component of D, we have that

$$\mathrm{mult}_x (D_i|_H) = \mathrm{mult}_x (D_i).$$

Also, we assume H is contained in no D_i.

Now, we will set $D_H = D|_H$.

By the previous assumption we have $\mathrm{mult}_x(D_H) = \mathrm{mult}_x(D) < 1$.

By induction $\mathcal{J}(H, D_H)_x = \mathcal{O}_{H,x}$.

By inversion of adjunction, we conclude that $\mathcal{J}(X, D)_x = \mathcal{O}_{X,x}$.

$\square$

Proposition: In the setting of the restriction theorem.

For any number $0 < s < 1$, we have that

$$\mathcal{J}(X, D + (1-t)H)_H \subseteq \mathcal{J}(H, (1-s)D_H).$$

for all sufficiently small t.

Remark: for t small enough and $0 < s < 1$.

$$\begin{array}{ccc} \mathcal{J}(X, D + (1-t)H)_H \subseteq & \mathcal{J}(H, (1-s)D_H) & \\ & \cup\!\!| & \\ & \mathcal{J}(H, D_H) & \subseteq \mathcal{J}(X, D)_H. \end{array}$$

Proof of prop: $E \subseteq X'$ different from H' which is contained in the support $K_{X'/X} + \mu^*(D+H)$ that meets H'.

Write $\bar{E} = E \cap H'$.

It is enough to show

$$\mathrm{ord}_E([\mu^*((1-t)H + D)] - K_{X'/X}) \geq$$

$$\mathrm{ord}_{\bar{E}}([\mu_H^*((1-s)D_H)] - K_{H'/H}). \qquad \text{(*)}$$

holds whenever the right side is positive

$b = \mathrm{ord}_E(K_{X'/X})$, $a = \mathrm{ord}_E(\mu^* H)$ $r = \mathrm{ord}_E(\mu^* D)$

$r > 0$ otherwise the right side of (*) is negative

By adjunction, $\mathrm{ord}_{\bar{E}}(K_{H'/H}) = b - a$.

Proving (*) turns down to prove.

$$[(1-t)a + r] - b \geq [(1-s)r] - (b-a).$$

This holds whenever $t \leq \frac{sr}{\underset{\substack{\vee \\ 0}}{a}}$ $\square$.

Corollary: Fix a number $s \in (0,1)$. Then

$$\mathcal{J}(X, D + (1-t)H)_H \subseteq \mathcal{J}(H, (1-s)D_H)$$

for every t small enough.

In particular, if $(X, D + (1-t)H)$ is klt, then so does $(H, (1-s)D_H)$.

Theorem (Restriction on singular varieties):
(X, Δ) log pair. $H \subseteq X$ reduced integral Cartier divisor with $H \not\subseteq \operatorname{Supp} \Delta$. Assume H is a normal variety. $D \subseteq X$ effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X whose support does not contain H. Then

$$\mathcal{J}((H, \Delta_H); D_H) \subseteq \mathcal{J}((X, \Delta); D)_H.$$

Remark: Let X be a complex variety and $\mathcal{I}$ an ideal sheaf on X. The multiplier ideal $\mathcal{J}(\mathcal{I}^c)$ is the ideal sheaf generated by all functions h such that

$$\frac{|h|^2}{\sum_i^r |f_i|^{2c}}$$

is locally integrable, where the f_i's is a finite set of local generators of $\mathcal{I}$.

This gives a natural inclusion $\mathfrak{a} \subseteq \mathcal{J}(\mathfrak{a})$.

Question: $D_1, D_2 \subseteq X$, is there a way to compare $\mathcal{J}(X, D_1 + D_2)$ with $\mathcal{J}(X, D_1)$ and $\mathcal{J}(X, D_2)$?

Example: $X = \mathbb{C}^2_{x,y}$, $D_1 = \langle x \rangle$, $D_2 = \langle y \rangle$.

$\mathcal{J}(X, D_1 + D_2) = \langle xy \rangle$,

$\mathcal{J}(X, D_1) = \langle x \rangle$

$\mathcal{J}(X, D_2) = \langle y \rangle$

$$\mathcal{J}(X, D_1 + D_2) = \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2)$$

Example: $X = \mathbb{C}^2$, $D_1 = \frac{1}{2}\langle x \rangle$ $D_2 = \frac{1}{2}\langle x \rangle$.

$$\mathcal{J}(X, D_1) = \mathcal{O}_X.$$

$$\mathcal{J}(X, D_2) = \mathcal{O}_X.$$

$$\mathcal{J}(X, D_1 + D_2) = \mathcal{J}(\mathbb{C}^2, \langle x \rangle) = \langle x \rangle.$$

Theorem (Subadditivity): X a smooth variety, $D_1, D_2 \overset{\geq 0}{\subseteq} X$

(i) $\mathcal{J}(X, D_1 + D_2) \subseteq \mathcal{J}(X, D_1)\,\mathcal{J}(X, D_2)$

(ii) $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_X$ ideal sheaves, then

$(D_i, D_1 \cap D_2)$ klt $\Downarrow$ eq ??

$$\mathcal{J}(\mathfrak{a}^c \mathfrak{b}^d) \subseteq \mathcal{J}(\mathfrak{a}^c)\,\mathcal{J}(\mathfrak{b}^d).$$

for any $c, d > 0$. In particular $\mathcal{J}(\mathfrak{a} \cdot \mathfrak{b}) \subseteq \mathcal{J}(\mathfrak{a})\,\mathcal{J}(\mathfrak{b})$.

Notation: $\mu_i : X_i' \longrightarrow X_i$ log resolution of (X_i, D_i)

$$\begin{tikzcd}
X_1' \arrow[d, "\mu_1"'] & X_1' \times X_2' \arrow[l, "q_1"'] \arrow[r, "q_2"] \arrow[d, "\mu_1 \times \mu_2"] & X_2' \arrow[d, "\mu_2"] \\
X_1 & X_1 \times X_2 \arrow[l, "p_1"] \arrow[r, "p_2"'] & X_2
\end{tikzcd}$$

Lemma: The product $\mu_1 \times \mu_2 : X_1' \times X_2' \longrightarrow X_1 \times X_2$ is a log resolution of $(X_1 \times X_2,\ p_1^* D_1 + p_2^* D_2)$.

Proposition: There is an equality

$$\mathcal{J}(X_1 \times X_2,\ p_1^* D_1 + p_2^* D_2) = p_1^{-1} \mathcal{J}(X_1, D_1) \cdot p_2^{-1} \mathcal{J}(X_2, D_2)$$

Proof: $\mathcal{J}(X_1 \times X_2,\ p_1^* D_1 + p_2^* D_2) =$

log smooth

$$(\mu_1 \times \mu_2)_* \mathcal{O}_{X_1' \times X_2'}\left(K_{X_1' \times X_2' / X_1 \times X_2} - [(\mu_1 \times \mu_2)^* (p_1^* D_1 + p_2^* D_2)]\right) =$$

Kunneth

$$(\mu_1 \times \mu_2)_* \Big(q_1^* \mathcal{O}_{X_1'}(K_{X_1'/X_1} - [\mu_1^* D_1]) \otimes q_2^* \mathcal{O}_{X_2'}(K_{X_2'/X_2} - [\mu_2^* D_2]) \Big) =$$

def of $\mathcal{J}$.

$$p_1^* \mu_{1*} \mathcal{O}_{X_1'}(K_{X_1'/X_1} - [\mu_1^* D_1]) \otimes p_2^* \mu_{2*} \mathcal{O}_{X_2'}(K_{X_2'/X_2} - [\mu_2^* D_2]) =$$

flatness of p_1 and p_2

$$p_1^* \mathcal{J}(X_1, D_1) \otimes p_2^* \mathcal{J}(X_2, D_2) =$$

$$p_1^{-1} \mathcal{J}(X_1, D_1) \cdot p_2^{-1} \mathcal{J}(X_2, D_2)$$

$\square$.

Proof of subaddtivity:

X quasi-projective. Take $X_1 = X_2 = X$.

$$\begin{tikzcd} \Delta \cong X \subseteq X \times X \arrow[d, "p_1"'] \arrow[dr, "p_2"] & \\ X & X \end{tikzcd}$$

the diagonal embedding.

$$\mathcal{J}(X, D_1 + D_2) = \mathcal{J}(\Delta, (p_1^* D_1 + p_2^* D_2)_\Delta)$$

$$\subseteq \mathcal{J}(X \times X, p_1^* D_1 + p_2^* D_2)_\Delta$$

rest thm

$$\mathcal{J}(X \times X, p_1^* D_1 + p_2^* D_2)_\Delta = \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2).$$

$\square$

Topics:
- Singularities of Θ divisors
- Summation Theorem.